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Fuzzy Supersphere and Supermonopole

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Abstract

It is well-known that coordinates of a charged particle in a monopole background become non-commutative. In this paper, we study the motion of a charged particle moving on a supersphere in the presence of a supermonopole. We construct a supermonopole by using a supersymmetric extension of the first Hopf map. We investigate algebras of angular momentum operators and supersymmetry generators. It is shown that coordinates of the particle are described by fuzzy supersphere in the lowest Landau level. We find that there exist two kinds of degenerate wavefunctions due to the supersymmetry. Ground state wavefunctions are given by the Hopf spinor and we discuss their several properties.

1 Introduction

Over the past few years several papers have been devoted to the study of a relationship between noncommutative geometry and string theory. The need of noncommutative geometry in string theory is easily understood by considering a world-volume action of D-branes. D-branes are defined as the endpoints of open strings. Since gauge fields appear in the ground state of open strings, the low energy dynamics of D-branes is described by gauge fields. One of the most interesting aspects is the appearance of nonabelian gauge symmetry from the world-volume theory of some coincident D-branes, and transverse coordinates of N D-branes are expressed by $U(N)$ adjoint scalars. The appearance of the matrix-valued coordinates implies a relationship between string theory and noncommutative geometry.

The appearance of noncommutative geometry in string theory can be understood from a different point of view. It is also observed that a world volume theory on a D-brane in the presence of NS-NS two form background is described by noncommutative Yang-Mills theory [1]. We can say that noncommutative geometry appears in two different situations. A D2-brane can be constructed from multiple D0-branes by imposing a noncommutative relation on their coordinates. The size of matrix represents the number of D0-branes. On the other hand, world volume coordinates of a D2-brane under the strong magnetic field become noncommutative. The magnetic charge is interpreted as the number of D0-branes. These two descriptions are supposed to be same. As these examples show, to study these two descriptions leads to understanding a relationship between D-branes with different dimensions.

The existence of these descriptions is easily understood by considering the quantum Hall system. It is well-known that noncommutative coordinates can be understood as guiding center coordinates in a strong magnetic field. The above two descriptions of D-branes are related to the existence of two kinds of coordinate, usual commutative coordinates and noncommutative guiding center coordinates. The appearance of noncommutative geometry in both theories is a common feature. By taking the lowest Landau limit or the zero slope limit (discussed in [1]), both theories obtain effective descriptions in terms of noncommutative geometry. A proposal given in [2] manifests the fact that the quantum Hall system is described by string theories.

Another recent development in string theory is understanding of noncommutative superspace. If we consider string theories in the R-R field strength or graviphoton background, coordinates of superspace become non(anti)commutative [3, 4, 5]. Various aspects of noncommutative superspace have been studied in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22]. Some studies from the viewpoint of supermatrix models are found in [23, 24, 25].

As in the bosonic noncommutative geometry, it is important to investigate two descriptions of noncommutative superspace. In this paper, we consider the motion of a charged particle on a supersphere in a supermonopole background as a supersymmetric generalization of the quantum Hall system. We show a relationship between commutative coordinates and noncommutative guiding center coordinates. A noncommutative version of supersphere called fuzzy supersphere has been investigated in [26, 27, 28]. We expect that such a noncommutative space arises in the lowest Landau level. The reason for dealing with a (fuzzy) sphere is that the quantity such as the charge of D0-branes is given by a finite quantity. A noncommutative sphere is usually obtained

by introducing a cut-off parameter for the angular momentum in a usual sphere. It is introduced as a monopole charge in the context of the quantum Hall system. The cut-off parameter is related to the number of D0-branes (quanta); therefore it can be finite for compact spaces. This is an advantage in order to compare two descriptions. The realization of noncommutative superspace in the lowest Landau level has also been reported in [11, 29].

The organization of this paper is as follows. We first review the (bosonic) two-sphere system in section 2. The Dirac monopole is introduced by the first Hopf map. According to the Hopf map, the gauge field is obtained from the so-called Hopf spinor. The Hopf spinor plays an important role in the quantum Hall system since it becomes a ground state eigenfunction of the Hamiltonian. We explain how a noncommutative space arises after we take the strong magnetic field limit. In section 3, we introduce a supersymmetric generalization of the Dirac monopole by using a supersymmetric generalization of the first Hopf map. The construction of the supermonopole is based on the method given in [30]. We explicitly construct the Hopf spinors for an arbitrary monopole charge. In section 4, we analyze the motion of a particle moving on $S^{2,2}$. Symmetries of $S^{2,2}$ are given by Lie supergroup $OSp(1|2)$. The Hamiltonian of a free particle is written down in terms of the $osp(1|2)$ (and $osp(2|2)$) generators. The contribution of the monopole is added by replacing usual derivatives with gauge covariant derivatives. The $osp(1|2)$ generators in the monopole background are deformed compared to those without the monopole background. We can obtain guiding center coordinates from the deformed $osp(1|2)$ generators. It is shown that commutative coordinates of a particle are identified with noncommutative guiding center coordinates in the lowest Landau level. They are found to satisfy the algebra of the fuzzy supersphere. Ground state wavefunctions are obtained from the Hopf spinors. We have two kinds of wavefunctions with the same energy because of the supersymmetry. We discuss their probability density and transformation property under the supersymmetry. Section 5 is devoted to summary and discussions. Notations related to the superalgebra are summarized in appendix A. In appendix B, we comment on the $osp(2|2)$ algebra. The $osp(2|2)$ generators are constructed from the $osp(1|2)$ generators and play an important role in constructing the Hamiltonian. We show how they are deformed in the presence of the supermonopole. The representation theory of $OSp(1|2)$ and $OSp(2|2)$ is reviewed in appendix C. The detailed calculation of (54) is presented in appendix D.

2 Review of two-sphere system

In this section, we review a (bosonic) two-sphere system. We consider a particle moving on a two-sphere in the background of a monopole put at the origin.

Let us first introduce the Dirac monopole based on the first Hopf map. The first Hopf map is defined as a map from S^3 to S^2 which is expressed as

$$x_i = 2r\phi^\dagger\sigma_{(1/2)i}\phi, \quad (1)$$

where $\sigma_{(1/2)i}$ is the spin 1/2 representation of $su(2)$ ¹. ϕ is a complex two-components spinor satisfying $\phi^\dagger\phi = 1$ and is called Hopf spinor. ϕ^\dagger means the hermitian conjugate of ϕ . The condition

¹It is related to the Pauli matrix as $2\sigma_{(1/2)i} = \sigma_i$

$\phi^\dagger \phi = 1$ leads to $x_i x_i = r^2$. The Hopf spinor satisfying (1) is explicitly given by

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{2r(r+x_3)}} \begin{pmatrix} r+x_3 \\ x_1+ix_2 \end{pmatrix} e^{i\chi}, \quad (2)$$

where $e^{i\chi}$ is a $U(1)$ phase. A $U(1)$ gauge transformation is generated by $\chi \rightarrow \chi + \Lambda$. A $U(1)$ gauge field is obtained from the Hopf spinor as

$$A_i dx_i = -i \frac{\hbar}{e} \phi^\dagger d\phi = -\frac{g}{r(r+x_3)} \epsilon_{ij3} x_j dx_i, \quad (3)$$

where $g \equiv \hbar/2e$ is the monopole charge. A monopole with $g = \hbar S/e$ is obtained by replacing ϕ with the following $(2S+1)$ -components spinor:

$$\phi_{(S,m)} = \sqrt{\frac{(2S)!}{(S-m)!(S+m)!}} \phi_1^{S+m} \phi_2^{S-m}, \quad (4)$$

where $2S$ is a positive integer, and m takes values $-S, -S+1, \dots, S$. The $S = 1/2$ case corresponds to (2). The normalization is determined from the following condition,

$$\int_{S^2} d\Omega \phi_{(S,m)}^* \phi_{(S,m')} = \frac{4\pi}{2S+1} \delta_{m,m'}. \quad (5)$$

The equation (1) is replaced with

$$x_i = \frac{1}{S} r \phi_{(S)}^\dagger \sigma_{(S)i} \phi_{(S)}, \quad (6)$$

where $\sigma_{(S)i}$ is the spin S representation of $su(2)$. This x_i also satisfies $x_i x_i = r^2$. We note that this construction naturally realizes the Dirac quantization condition:

$$eg = \hbar S. \quad (7)$$

The field strength of this monopole is

$$F_{ij} = \partial_i A_j - \partial_j A_i = \frac{g}{r^3} \epsilon_{ijk} x_k. \quad (8)$$

The first Chern number is calculated as

$$c_1 = \frac{e}{2\pi\hbar} \int_{S^2} F = 2S. \quad (9)$$

We next investigate the motion of a charged particle moving on a two-sphere in the monopole background. The Hamiltonian of such a particle is given by

$$H = \frac{1}{2mr^2} \Lambda_i \Lambda_i, \quad (10)$$

where m is the mass of the particle, and Λ_i is the orbital angular momentum of the charged particle in the monopole background:

$$\Lambda_i = \epsilon_{ijk} x_j (-i\hbar \partial + eA)_k. \quad (11)$$

These Λ_i no longer satisfy the algebra of the usual angular momentum and are deformed to

$$[\Lambda_i, \Lambda_j] = i\hbar\epsilon_{ijk} \left(\Lambda_k - \frac{eg}{r}x_k \right). \quad (12)$$

Operators generating the $SU(2)$ rotation in the presence of the monopole are found to be

$$L_i = \Lambda_i + \frac{eg}{r}x_i. \quad (13)$$

The last term represents the contribution from the monopole background, and L_i can be interpreted to be the total angular momentum. They actually satisfy

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k, \quad [L_i, \Lambda_j] = i\hbar\epsilon_{ijk}\Lambda_k, \quad [L_i, x_j] = i\hbar\epsilon_{ijk}x_k. \quad (14)$$

From these relations, it is easily shown that $[L_i, H] = 0$, which manifests the fact that this system has the $SU(2)$ symmetry generated by L_i . We suppose the representation of L_i to be the spin l . Then by using the following relation

$$\Lambda_i\Lambda_i = L_iL_i - (eg)^2 = \hbar^2 \left(l(l+1) - S^2 \right), \quad (15)$$

we can get the following energy eigenvalue of the Hamiltonian,

$$E_n = \frac{\hbar^2}{2mr^2} (n(n+1) + (2n+1)S), \quad (16)$$

where we have set $l = n + S$ ($n = 0, 1, 2, \dots$). n plays the role of the Landau level index and $n = 0$ corresponds to the lowest Landau level. Since an energy interval between the lowest Landau level and the first Landau level is given by $\Delta E = S\hbar^2/mr^2$, the motion of the particle is confined to the lowest Landau level in the strong magnetic field limit:

$$S/mr \gg 1. \quad (17)$$

The degeneracy of the lowest Landau level is $2S + 1$. It is related to the size of noncommutative space as we will see later.

As in the well-known planar system, the motion of the charged particle obeys the cyclotron motion. The guiding center coordinates X_i can be introduced as

$$X_i = \alpha L_i, \quad \alpha \equiv \frac{r}{eg}. \quad (18)$$

They satisfy the following noncommutative relation

$$[X_i, X_j] = i\hbar\alpha\epsilon_{ijk}X_k. \quad (19)$$

From the equation (13), we obtain a relationship between the guiding center coordinates and the commutative coordinates as

$$X_i = \alpha\Lambda_i + x_i. \quad (20)$$

The radius of the cyclotron motion in the n -th Landau level is evaluated as

$$r_n^{cyc} = \alpha\hbar\sqrt{n(n+1) + (2n+1)S}. \quad (21)$$

In the lowest Landau level, it becomes

$$r_0^{cyc} = \alpha \hbar \sqrt{S} = \frac{r}{\sqrt{S}}. \quad (22)$$

Since the radius r_0^{cyc} becomes much smaller than r in the strong magnetic field limit (17), the commutative coordinates x_i are identified with the noncommutative coordinates X_i in the lowest Landau level. The noncommutative geometry described by X_i is known as fuzzy sphere. The radius of the cyclotron motion for the ground state provides the noncommutative length: $l_{NC} \equiv r_0^{cyc}$. The radius of the fuzzy sphere is given by the quadratic Casimir of $su(2)$ as

$$r^2 = \hbar^2 \alpha^2 S(S+1). \quad (23)$$

If we substitute $\alpha = r/eg$, the Dirac quantization condition (7) is reproduced in the large S limit.

We shall consider the thermodynamic limit. It is given by the large S limit with keeping the noncommutative scale l_{NC} finite. In this limit, the energy eigenvalue (16) approaches

$$E_n \rightarrow \frac{\hbar^2}{ml_{NC}^2} \left(n + \frac{1}{2} \right). \quad (24)$$

This corresponds to the planar Landau levels.

Before finishing this section, we comments on the eigenstates of this system. When $S = 1/2$, the Hopf spinor (2) is found to become the ground state wavefunction of the Hamiltonian. In general, the eigenstate with the eigenvalue E_0 in (16) is given by the Hopf spinor (4). It is because the Hopf spinor (2) (and (4)) transforms as an $SU(2)$ spinor. We should notice that the conjugate spinor $\bar{\phi}$ does not enter the eigenstate in the lowest Landau level. This fact is an analogous to the result in the planar system where wavefunctions in the lowest Landau level are written in terms of polynomials of only z (up to a Gaussian factor). The probability density of the eigenstates is given by

$$|\phi_{(S,m)}|^2 = \frac{(2S)!}{(S-m)!(S+m)!} \left(\frac{1}{2r} \right)^{2S} (r+x_3)^{S+m} (r-x_3)^{S-m}. \quad (25)$$

This state forms a ring and is localized at $x_3 = (m/S)r$. This result reminds us of the planar system in the symmetric gauge.

3 Supermonopole

In the previous section, we reviewed the bosonic two-sphere system and observed that coordinates of a charged particle are described by the fuzzy two-sphere in the lowest Landau level. In the following sections, we study the motion of a charged particle moving on the supersphere $S^{2,2}$ as a supersymmetric generalization of the previous section.² We expect that the coordinates are described by the fuzzy supersphere in the same way as the bosonic case.

²A similar analysis for the superspace $SU(1|2)/[U(1) \times U(1)]$ has been made in [29].

We first review the supersphere. The supersphere $S^{2,2}$ is characterized by the coset space given by $OSp(1|2)/U(1)$. Let x_i ($i = 1, 2, 3$) and θ_α ($\alpha = 1, 2$) be coordinates of the supersphere which are related as

$$x_i x_i + C_{\alpha\beta} \theta_\alpha \theta_\beta = r^2, \quad (26)$$

where $C_{\alpha\beta}$ is the antisymmetric tensor with $C_{12} = 1$. We define a coordinate y_i such that $y_i y_i = r^2$. A space which is defined by the coordinate y_i is called the *body* of the superspace. Hence S^2 is the body of $S^{2,2}$. It is related to x_i as

$$y_i = \left(1 + \frac{\theta C \theta}{2r^2}\right) x_i. \quad (27)$$

The remaining coordinate θ_α is called the *soul*.

The supersphere has an $SU(2)$ rotational symmetry and supersymmetry which are generated by

$$\begin{aligned} J_i &= -i\hbar \epsilon_{ijk} x_j \partial_k + \frac{1}{2} \hbar \theta_\alpha (\sigma_i)_{\alpha\beta} \partial_\beta, \\ J_\alpha &= \frac{1}{2} \hbar x_i (C \sigma_i)_{\alpha\beta} \partial_\beta - \frac{1}{2} \hbar \theta_\beta (\sigma_i)_{\beta\alpha} \partial_i, \end{aligned} \quad (28)$$

respectively. They satisfy the following $osp(1|2)$ algebra,

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k, \quad [J_i, J_\alpha] = \frac{1}{2} \hbar (\sigma_i)_{\beta\alpha} J_\beta, \quad \{J_\alpha, J_\beta\} = \frac{1}{2} \hbar (C \sigma_i)_{\alpha\beta} J_i. \quad (29)$$

The $osp(1|2)$ algebra is simply reviewed in appendix C. The coordinates transform under the supersymmetry as

$$\begin{aligned} \delta x_i &= \frac{1}{2} (\epsilon \sigma_i C \theta), \\ \delta \theta_\alpha &= -\frac{1}{2} (\epsilon \sigma_i)_\alpha x_i, \end{aligned} \quad (30)$$

where ϵ_α are Grassmann parameters. The radius of $S^{2,2}$ is invariant under the supersymmetry

$$\delta r = 0. \quad (31)$$

Let us next introduce a supersymmetric generalization of the Dirac monopole. We use a supersymmetric generalization of the first Hopf map $S^{3,2} \rightarrow S^{2,2}$ based on [30]. We will obtain an explicit form of the Hopf spinor expressed by the coordinate of $S^{2,2}$. It plays an important role since it becomes a wavefunction in the lowest Landau level as is discussed in the next section. The map is expressed by

$$x_i = 2r \phi^\dagger l_i \phi, \quad \theta_\alpha = 2r \phi^\dagger v_\alpha \phi, \quad (32)$$

where

$$l_i = \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad v_1 = \frac{1}{2} \begin{pmatrix} 0 & \eta \\ \xi^T & 0 \end{pmatrix}, \quad v_2 = \frac{1}{2} \begin{pmatrix} 0 & \xi \\ -\eta^T & 0 \end{pmatrix} \quad (33)$$

are the three dimensional representation of $osp(1|2)$, and

$$\eta = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \quad (34)$$

ϕ is a complex three-components spinor which satisfies

$$\phi^\dagger \phi = 1, \quad (35)$$

where ϕ^\dagger is defined as $(\phi_1^*, \phi_2^*, -\psi^*)$. It must be noted that the minus sign is added to the third component. An explicit form of ϕ is given by the coordinates of $S^{2,2}$ as

$$\begin{aligned} \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \psi \end{pmatrix} &= \frac{1}{\sqrt{2r^3(r+x_3)}} \begin{pmatrix} (r+x_3) \left(r - \frac{1}{4(r+x_3)} \theta C \theta \right) \\ (x_1 + ix_2) \left(r + \frac{1}{4(r+x_3)} \theta C \theta \right) \\ -((r+x_3)\theta_1 + (x_1 + ix_2)\theta_2) \end{pmatrix} e^{i\chi} \\ &= \frac{1}{\sqrt{2r^3(r+y_3)}} \begin{pmatrix} (r+y_3) \left(r - \frac{1}{4r} \theta C \theta \right) \\ (y_1 + iy_2) \left(r - \frac{1}{4r} \theta C \theta \right) \\ -((r+y_3)\theta_1 + (y_1 + iy_2)\theta_2) \end{pmatrix} e^{i\chi} \end{aligned} \quad (36)$$

where χ is a bosonic coordinate and $e^{i\chi}$ is a $U(1)$ phase factor. A $U(1)$ local gauge transformation is induced by $\chi \rightarrow \chi + \Lambda$. From this explicit representation, the equation (35) is checked by making use of (70). A $U(1)$ gauge field is obtained from the Hopf spinor ϕ as

$$A_i dx_i + A_\alpha d\theta_\alpha = -i \frac{\hbar}{e} \phi^\dagger d\phi. \quad (37)$$

Hence each component of A is obtained as

$$\begin{aligned} A_i &= -\frac{g}{r(r+x_3)} \epsilon_{ij3} x_j \left(1 + \frac{2r+x_3}{2r^2(r+x_3)} \theta C \theta \right) \\ &= -\frac{g}{r(r+y_3)} \epsilon_{ij3} y_j \left(1 + \frac{1}{2r^2} \theta C \theta \right), \\ A_\alpha &= \frac{ig}{r^3} x_i (\theta \sigma_i C)_\alpha \\ &= \frac{ig}{r^3} y_i (\theta \sigma_i C)_\alpha, \end{aligned} \quad (38)$$

where $g \equiv \hbar/2e$ is the monopole charge. Note that A satisfies the reality condition $A^\dagger = A$. This gauge field is singular at the south pole. We can construct the gauge field which is singular at the north pole by using the following Hopf spinor:

$$\phi' = \begin{pmatrix} \phi'_1 \\ \phi'_2 \\ \psi' \end{pmatrix} = \frac{1}{\sqrt{2r^3(r-x_3)}} \begin{pmatrix} (x_1 - ix_2) \left(r + \frac{1}{4(r-x_3)} \theta C \theta \right) \\ (r-x_3) \left(r - \frac{1}{4(r-x_3)} \theta C \theta \right) \\ -((x_1 - ix_2)\theta_1 + (r-x_3)\theta_2) \end{pmatrix} e^{i\chi}. \quad (39)$$

The corresponding gauge field is

$$\begin{aligned} A'_i &= \frac{g}{r(r-x_3)} \epsilon_{ij3} x_j \left(1 + \frac{2r-x_3}{2r^2(r-x_3)} \theta C \theta \right), \\ A'_\alpha &= \frac{ig}{r^3} x_i (\theta \sigma_i C)_\alpha. \end{aligned} \quad (40)$$

(38) and (40) are related by the gauge transformation such that the gauge parameter is given by $\tan \Lambda = x_2/x_1$. A monopole which has a larger charge is obtained by using the following Hopf spinor,

$$\begin{aligned}\Phi_{(S,m)} &= \sqrt{\frac{(2S)!}{(S-m)!(S+m)!}} \phi_1^{S+m} \phi_2^{S-m}, \\ \Psi_{(S,m')} &= \sqrt{\frac{(2S)!}{(S-1/2-m')!(S-1/2+m')!}} \phi_1^{S-1/2+m'} \phi_2^{S-1/2-m'} \psi,\end{aligned}\quad (41)$$

where m runs over $-S, -S+1, \dots, S$, and m' over $-S+1/2, -S+3/2, \dots, S-1/2$. The orthonormal relations are

$$\begin{aligned}\int_{S^{2,2}} d\Omega_{(2,2)} \Phi_{(S,m)}^* \Phi_{(S,m')} &= \frac{8\pi S}{2S+1} \delta_{m,m'} \\ \int_{S^{2,2}} d\Omega_{(2,2)} \Psi_{(S,m)}^* \Psi_{(S,m')} &= 4\pi \delta_{m,m'} \\ \int_{S^{2,2}} d\Omega_{(2,2)} \Phi_{(S,m)}^* \Psi_{(S,m')} &= 0,\end{aligned}\quad (42)$$

where we have defined $d\Omega_{(2,2)} = d\Omega_{S^2} d\theta_1 d\theta_2$. In this case, the relation (32) is modified to

$$x_i = \frac{1}{S} r \phi_{(S)}^\dagger l_{(S)i} \phi_{(S)}, \quad \theta_\alpha = \frac{1}{S} r \phi_{(S)}^\dagger v_{(S)\alpha} \phi_{(S)}, \quad (43)$$

where $\phi_{(S)} \equiv (\Phi_{(S)}, \Psi_{(S)})^T$, and $l_{(S)i}$ and $v_{(S)\alpha}$ are $(4S+1)$ -dimensional representation of $osp(1|2)$. The gauge field strength is calculated as

$$\begin{aligned}F = dA &= \frac{1}{2} F_{ij} dx_i \wedge dx_j + F_{i\alpha} dx_i \wedge d\theta_\alpha + \frac{1}{2} F_{\alpha\beta} d\theta_\alpha \wedge d\theta_\beta \\ &= \frac{1}{2} (\partial_i A_j - \partial_j A_i) dx_i \wedge dx_j + (\partial_i A_\alpha - \partial_\alpha^R A_i) dx_i \wedge d\theta_\alpha \\ &\quad + \frac{1}{2} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) d\theta_\alpha \wedge d\theta_\beta,\end{aligned}\quad (44)$$

where we have used the notation such as $\partial_\alpha A = (\partial/\partial\theta_\alpha)A$ and $\partial_\alpha^R A = \partial A/\partial\theta_\alpha$. Hence we get

$$\begin{aligned}F_{ij} &= \frac{g}{r^3} \epsilon_{ijk} x_k \left(1 + \frac{3}{2r^2} \theta C \theta\right) \\ &= \frac{g}{r^3} \epsilon_{ijk} y_k \left(1 + \frac{1}{r^2} \theta C \theta\right), \\ F_{i\alpha} &= -\frac{2g}{r^3} i \left(\frac{3}{2} \frac{x_i x_j}{r^2} - \frac{1}{2} \delta_{ij}\right) (\theta \sigma_j C)_\alpha \\ &= -\frac{2g}{r^3} i \left(\frac{3}{2} \frac{y_i y_j}{r^2} - \frac{1}{2} \delta_{ij}\right) (\theta \sigma_j C)_\alpha, \\ F_{\alpha\beta} &= i \frac{2g}{r^3} x_i (\sigma_i C)_{\alpha\beta} \left(1 + \frac{3}{2r^2} \theta C \theta\right) \\ &= i \frac{2g}{r^3} y_i (\sigma_i C)_{\alpha\beta} \left(1 + \frac{1}{r^2} \theta C \theta\right),\end{aligned}\quad (45)$$

where the monopole charge is $g = \hbar S/e$.

We next see how the above components transform under the supersymmetry (30). Defining the bosonic part of the magnetic field as

$$B_i \equiv \frac{g}{r^3} x_i \left(1 + \frac{3}{2r^2} \theta C \theta \right), \quad (46)$$

we obtain

$$\begin{aligned} \delta B_i &= -i \epsilon_\alpha F_{i\alpha}, \\ \delta F_{i\alpha} &= -\frac{1}{4} \epsilon_{ijk} (\epsilon \sigma_k C)_\alpha B_j + \frac{i}{2} (\epsilon C)_\alpha B_i. \end{aligned} \quad (47)$$

We can recognize that B_i and $F_{i\alpha}$ form a multiplet under the supersymmetry (30).

Let us calculate the first Chern character of the supermonopole [30]. We define it as

$$\begin{aligned} c_1 &= \frac{e}{2\pi\hbar} \int_{S^{2,2}} F \\ &= \frac{e}{2\pi\hbar} \int_{S^{2,2}} \left(\frac{1}{2} F_{ij} dx_i \wedge dx_j + F_{i\alpha} dx_i \wedge d\theta_\alpha + \frac{1}{2} F_{\alpha\beta} d\theta_\alpha \wedge d\theta_\beta \right). \end{aligned} \quad (48)$$

The important point is that the coordinate x_i depends on the Grassmann coordinates due to the relation (26):

$$\sqrt{x_i x_i} = \sqrt{r^2 - \theta C \theta} = r - \frac{1}{2r} \theta C \theta. \quad (49)$$

The integration over the Grassmann variables is evaluated by the Berezin integral. It is found that the second and third terms in (48) vanish by integrating the Grassmann coordinates. As for the first term, the dependence of the Grassmann coordinates in F_{ij} cancels by that in $dx_i \wedge dx_j$ which comes from (49). Consequently the integral over the supersphere results in the integral over the body:

$$c_1 = \frac{e}{2\pi\hbar} \int_{S^2} \frac{1}{2} F_{ij|_{\theta=0}} dy_i \wedge dy_j = 2S. \quad (50)$$

We have obtained the same result as the bosonic case (9).

4 Fuzzy supersphere as the lowest Landau level

In this section, we analyze the motion of a particle moving on $S^{2,2}$ in the presence of the supermonopole background and see how noncommutative superspace arises in the lowest Landau level.

The Hamiltonian we start with is the following ³,

$$H = \frac{1}{2mr^2} (\Lambda_i \Lambda_i + C_{\alpha\beta} \Lambda_\alpha \Lambda_\beta). \quad (51)$$

³This Hamiltonian does not provide a complete form of the kinetic term of a particle moving on $S^{2,2}$ though it is a supersymmetric generalization of the bosonic case. We, nevertheless, use this Hamiltonian since it is the simplest supersymmetric generalization and enables us to know some properties peculiar to supersymmetric systems. The *correct* Hamiltonian is given in appendix B.

Λ_i and Λ_α are the gauge covariant operators which are obtained from (28) by making the following replacements,

$$\begin{aligned}\partial_i &\rightarrow \partial_i + ieA_i, \\ \partial_\alpha &\rightarrow \partial_\alpha - ieA_\alpha.\end{aligned}\tag{52}$$

$(\Lambda_i, \Lambda_\alpha)$ are orthogonal to the coordinates (x_i, θ_α) :

$$x_i \Lambda_i + \theta C \Lambda = 0.\tag{53}$$

Since we have replaced the derivative with the gauge covariant derivatives, Λ_i and Λ_α no longer satisfy the $osp(1|2)$ algebra. Their commutation relations become

$$\begin{aligned}[\Lambda_i, \Lambda_j] &= i\hbar \epsilon_{ijk} \left(\Lambda_k - \frac{eg}{r} x_k \right), \\ [\Lambda_i, \Lambda_\alpha] &= \frac{1}{2} \hbar (\sigma_i)_{\beta\alpha} \left(\Lambda_\beta - \frac{eg}{r} \theta_\beta \right), \\ \{\Lambda_\alpha, \Lambda_\beta\} &= \frac{1}{2} \hbar (C\sigma_i)_{\alpha\beta} \left(\Lambda_i - \frac{eg}{r} x_i \right).\end{aligned}\tag{54}$$

The detailed derivation of these relations is shown in appendix D. Therefore, the $osp(1|2)$ generators in the supermonopole background are given by

$$\begin{aligned}L_i &\equiv \Lambda_i + \frac{1}{\alpha} x_i, \\ L_\alpha &\equiv \Lambda_\alpha + \frac{1}{\alpha} \theta_\alpha,\end{aligned}\tag{55}$$

where $\alpha \equiv r/eg = r/\hbar S$. They satisfy

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k, \quad [L_i, L_\alpha] = \frac{1}{2} \hbar (\sigma_i)_{\beta\alpha} L_\beta, \quad \{L_\alpha, L_\beta\} = \frac{1}{2} \hbar (C\sigma_i)_{\alpha\beta} L_i,\tag{56}$$

and

$$\begin{aligned}[L_i, \Lambda_j] &= i\hbar \epsilon_{ijk} \Lambda_k, \quad [L_i, x_j] = i\hbar \epsilon_{ijk} x_k, \\ [L_i, \Lambda_\alpha] &= \frac{1}{2} \hbar (\sigma_i)_{\beta\alpha} \Lambda_\beta, \quad [L_i, \theta_\alpha] = \frac{1}{2} \hbar (\sigma_i)_{\beta\alpha} \theta_\beta \\ [L_\alpha, \Lambda_i] &= -\frac{1}{2} \hbar (\sigma_i)_{\beta\alpha} \Lambda_\beta, \quad [L_\alpha, x_i] = -\frac{1}{2} \hbar (\sigma_i)_{\beta\alpha} \theta_\beta, \\ \{L_\alpha, \Lambda_\beta\} &= \frac{1}{2} \hbar (C\sigma_i)_{\alpha\beta} \Lambda_i, \quad \{L_\alpha, \theta_\beta\} = \frac{1}{2} \hbar (C\sigma_i)_{\alpha\beta} x_i.\end{aligned}\tag{57}$$

They also satisfy

$$x_i L_i + \theta C L = \hbar r S.\tag{58}$$

Let us now suppose that L_i and L_α belong to the superspin l representation of $OSp(1|2)$ whose dimension is $N = 4l + 1$. The quadratic Casimir is given by

$$L_i L_i + C_{\alpha\beta} L_\alpha L_\beta = \hbar^2 l \left(l + \frac{1}{2} \right).\tag{59}$$

We then have

$$\begin{aligned}\Lambda_i \Lambda_i + C_{\alpha\beta} \Lambda_\alpha \Lambda_\beta &= L_i L_i + C_{\alpha\beta} L_\alpha L_\beta - \hbar^2 S^2 \\ &= \hbar^2 \left(l \left(l + \frac{1}{2} \right) - S^2 \right),\end{aligned}\tag{60}$$

where we have used the equation (58). Using this equation, the energy eigenvalue of the Hamiltonian is found to be

$$E_n = \frac{\hbar^2}{2mr^2} \left(n \left(n + \frac{1}{2} \right) + \left(2n + \frac{1}{2} \right) S \right),\tag{61}$$

where we have set $l = n + S$ ($n = 0, 1, \dots$). The integer n characterizes the Landau level. It can be shown that the Hamiltonian has the $osp(1|2)$ symmetry

$$[L_i, H] = [L_\alpha, H] = 0.\tag{62}$$

This means that there exist a degeneracy generated by L_i and L_α , which is related to the extension of a noncommutative superspace realized in the lowest Landau level (as will be seen later).

We define the guiding center coordinates as

$$\begin{aligned}X_i &= \alpha L_i = \alpha \Lambda_i + x_i, \\ \Theta_\alpha &= \alpha L_\alpha = \alpha \Lambda_\alpha + \theta_\alpha.\end{aligned}\tag{63}$$

Noncommutative geometry is obtained in the similar way to the bosonic system. The motion of the particle is confined to the lowest Landau level by taking the large S limit (17). The radius of the cyclotron motion in the n -th Landau level is now given by

$$r_n^{scyc} = \alpha \hbar \sqrt{n(n + 1/2) + (2n + 1/2)S}.\tag{64}$$

The radius in the ground state ($n = 0$) becomes much smaller than the radius of the supersphere r in the large S limit; accordingly the coordinates (x_i, θ_α) are identified with the noncommutative guiding center coordinates (X_i, Θ_α) . The coordinates are given by the superspin S representation of $OSp(1|2)$ and form the following algebra,

$$\begin{aligned}[X_i, X_j] &= i\alpha \hbar \epsilon_{ijk} X_k, \\ [X_i, \Theta_\alpha] &= \frac{1}{2} \alpha \hbar (\sigma_i)_{\beta\alpha} \Theta_\beta, \\ \{\Theta_\alpha, \Theta_\beta\} &= \frac{1}{2} \alpha \hbar (C\sigma_i)_{\alpha\beta} X_i.\end{aligned}\tag{65}$$

The superspin S representation of $OSp(1|2)$ is given by a $(4S + 1) \times (4S + 1)$ matrix and is decomposed into the spin S and $(S - 1/2)$ representation of $SU(2)$. L_i and L_α generate the $SU(2)$ rotation and supersymmetry respectively, acting on the noncommutative coordinates as

$$\begin{aligned}[L_i, X_j] &= i\hbar \epsilon_{ijk} X_k, \quad [L_i, \Theta_\alpha] = \frac{1}{2} \hbar (\sigma_i)_{\beta\alpha} \Theta_\beta, \\ [L_\alpha, X_i] &= -\frac{1}{2} \hbar (\sigma_i)_{\beta\alpha} \Theta_\beta, \quad \{L_\alpha, \Theta_\beta\} = \frac{1}{2} \hbar (C\sigma_i)_{\alpha\beta} X_i.\end{aligned}\tag{66}$$

The radius of the fuzzy supersphere is provided by

$$r^2 = X_i X_i + C_{\alpha\beta} \Theta_\alpha \Theta_\beta = \alpha^2 \hbar^2 S \left(S + \frac{1}{2} \right). \quad (67)$$

The thermodynamic limit is given by the large S limit with keeping noncommutative scale l_{NC} finite. In this limit, (61) becomes

$$E_n \rightarrow \frac{\hbar^2}{m l_{NC}^2} \left(n + \frac{1}{4} \right). \quad (68)$$

We find that the ground state energy is lower than that of the bosonic system (24). This would be explained by the supersymmetry.

We discuss the eigenfunctions in the lowest Landau level. The Hopf spinor (41) becomes the eigenfunctions in the lowest Landau level since it is an $OSp(1|2)$ spinor. We also note that conjugate spinors do not appear in their expressions. A novel aspect compared to the bosonic system is the existence of the supersymmetry. Hence we have two kinds of eigenstates with the same energy. We can explicitly confirm that they are related by the supersymmetry transformation. For the superspin $1/2$ states, we have

$$\begin{pmatrix} \delta\Phi \\ \delta\Psi \end{pmatrix} = \begin{pmatrix} \delta\phi_1 \\ \delta\phi_2 \\ \delta\psi \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\epsilon_2\psi \\ \epsilon_1\psi \\ \epsilon_2\phi_2 + \epsilon_1\phi_1 \end{pmatrix}, \quad (69)$$

where ϵ_α ($\alpha = 1, 2$) are Grassmann parameters. The probability density of these states is calculated as

$$\begin{aligned} |\phi_1|^2 &\equiv \phi_1^* \phi_1 = \frac{r + y_3}{2r} \left(1 - \frac{1}{2r^2} \theta C \theta \right), \\ |\phi_2|^2 &\equiv \phi_2^* \phi_2 = \frac{r - y_3}{2r} \left(1 - \frac{1}{2r^2} \theta C \theta \right), \\ |\psi|^2 &\equiv -\psi^* \psi = \frac{1}{2r^2} \theta C \theta. \end{aligned} \quad (70)$$

The first (second) state is localized at the north (south) pole, $y_3 = r$ ($-r$). The third one does not depend on the coordinates of the body. These results can be generalized to the case of the superspin S . The supersymmetry transformation is given by

$$\begin{aligned} \delta\Phi_{(S,m)} &= \frac{1}{2} \left(-\epsilon_2 \sqrt{S+m} \Psi_{(S,m-1/2)} + \epsilon_1 \sqrt{S-m} \Psi_{(S,m+1/2)} \right), \\ \delta\Psi_{(S,m')} &= \frac{1}{2} \left(\epsilon_2 \sqrt{S+1/2-m'} \Phi_{(S,m'-1/2)} + \epsilon_1 \sqrt{S+1/2+m'} \Phi_{(S,m'+1/2)} \right). \end{aligned} \quad (71)$$

The probability density for these two states is

$$\begin{aligned} |\Phi_{(S,m)}|^2 &= C_{(S,m)}^2 \left(\frac{1}{2r} \right)^{2S} (r + y_3)^{S+m} (r - y_3)^{S-m} \left(1 - \frac{S}{r^2} \theta C \theta \right), \\ |\Psi_{(S,m')}|^2 &= C_{(S,m')}^2 \left(\frac{1}{2r} \right)^{2S-1} (r + y_3)^{S-1/2+m'} (r - y_3)^{S-1/2-m'} \frac{1}{2r^2} \theta C \theta, \end{aligned} \quad (72)$$

where we denote the normalization factors in (41) by $C_{(S,m)}$ and $C_{(S,m')}$. Φ and Ψ form rings on the body and are localized at $y_3 = mr/S$ and $y_3 = m'r/(S - 1/2)$ ($S \neq 1/2$) respectively.

5 Summary and discussions

In this paper, we have considered the motion of a charged particle moving on a supersphere in the presence of a supermonopole. The supermonopole was constructed by the supersymmetric first Hopf map. This system is a supersymmetric generalization of the quantum Hall system on a bosonic two-sphere. We obtained a relationship between the commutative coordinates and the noncommutative guiding center coordinates. It was shown that they were identified in the lowest Landau level. The guiding center coordinates form the algebra of the fuzzy supersphere. We also obtained two kinds of ground state wavefunctions from the Hopf spinor. They have the same energy and are related by the supersymmetry.

We would like to comment on a relationship to the noncommutativity of D-branes. The fact that coordinates of a charged particle is described by noncommutative guiding center coordinates is related to the two descriptions of D-branes (which is simply explained in the second paragraph in the introduction). See [31] for the discussion of spherical D2-branes. The number of D0-branes is expressed by the size of matrix (or noncommutative coordinate) in the D0-brane's description. On the other hand, it is expressed by the first Chern number of a magnetic monopole from the viewpoint of a D2-brane. These two quantities are given by $2S + 1$ and $2S$ respectively for the bosonic spherical system reviewed in section 2. The agreement of these two quantities can be seen in the limit of large S , which implies the fact that the two systems provide the same descriptions in this limit. We expect that such a comparison can be done in the supersymmetric system though an interpretation of D-brane is not clear. We evaluated the Chern number in (50) and found that it was given by the contribution only from the body space. Therefore it gave the same value as the bosonic monopole. It can be compared with the body of the superalgebra. The $osp(1|2)$ superalgebra contains the $su(2)$ subalgebra whose representation is decomposed into the spin S and $S - 1/2$ representations. It is natural to regard the spin S representation of the $su(2)$ subalgebra as the body of the $osp(1|2)$ superalgebra. Thus the comparison in the supersymmetric system resulted in that in the bosonic system.

A new element compared to the bosonic system is the existence of the supersymmetry. We used a supersymmetric extension of the first Hopf map based on the supergroup $OSp(1|2)$. The supermonopole constructed from the map showed the supersymmetric structure (47). Since the noncommutativity is expected to stem from the monopole field strength, such a structure leads to a supersymmetric noncommutativity. We also obtained the ground state wavefunctions which were given by the Hopf spinor. Their supersymmetric structure is explicitly shown in (71). The supersymmetric structure is naturally included due to the use of the supergroup $OSp(1|2)$.

We conclude this section with a future problem. The extension to higher dimensional systems remains as an interesting problem. A bosonic higher dimensional quantum Hall system was first constructed in [32]. Further generalizations have been discussed in [33, 34, 35]. We have investigated a relationship between such bosonic higher dimensional systems and noncommutative geometry in [36, 37]. Since the appearance of noncommutative geometry in higher dimensional systems is different from two dimensional systems, it is important to study how noncommutative superspaces arise in higher dimensional supersymmetric systems.

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A Notation

In this section, we summarize some notations which are related to supermatrix (superalgebra). Let X be a supermatrix:

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (\text{A.1})$$

where A and D are even elements, while B and C are odd (Grassmann) elements. We define the superadjoint operation \dagger as

$$X^\dagger = \begin{pmatrix} A^\dagger & C^\dagger \\ -B^\dagger & D^\dagger \end{pmatrix}, \quad (\text{A.2})$$

where \dagger means the usual adjoint operation. The superadjoint operation on the $osp(1|2)$ generators is, therefore, given by

$$l_i^\dagger = l_i, \quad v_\alpha^\dagger = C_{\alpha\beta} v_\beta. \quad (\text{A.3})$$

We next define superstar $*$ which act on Grassmann numbers as

$$(\theta_i \theta_j)^* = \theta_i^* \theta_j^*, \quad \theta_i^{**} = -\theta_i. \quad (\text{A.4})$$

The action on bosonic numbers is the usual complex conjugation. It acts on Grassmann coordinates of $S^{2,2}$ as

$$\theta_\alpha^* = C_{\alpha\beta} \theta_\beta, \quad (\theta C \theta)^* = \theta C \theta. \quad (\text{A.5})$$

B $osp(2|2)$ algebra in supermonopole background

In this section, we investigate the $osp(2|2)$ algebra in the supermonopole background. It plays an important role in constructing the Hamiltonian (discussed in the latter part of this section) or gauge field theories [26, 25].

The generators of the $osp(2|2)$ algebra are given by those of the $osp(1|2)$ algebra as

$$\begin{aligned} J^\gamma &= -\frac{2}{r}(\theta C)_\beta J_\beta \\ &= \frac{1}{r} \hbar x_i (\theta \sigma_i)_\beta \frac{\partial}{\partial \theta_\beta}, \\ J_\alpha^d &= \frac{1}{r} (\sigma_i)_{\beta\alpha} (\theta_\beta J_i + x_i J_\beta) \\ &= -\frac{r}{2} \hbar \left(1 + \frac{\theta C \theta}{2r^2} \right) C_{\alpha\beta} \frac{\partial}{\partial \theta_\beta} - \frac{1}{2r} \hbar (\theta \sigma_i \sigma_j)_\alpha x_i \frac{\partial}{\partial x_j}. \end{aligned} \quad (\text{B.1})$$

A relationship between the $osp(1|2)$ algebra and the $osp(2|2)$ algebra is further explained in the appendix C. They satisfy the following algebra,

$$\begin{aligned} [J^\gamma, J_\alpha] &= \hbar J_\alpha^d, \quad [J^\gamma, J_\alpha^d] = \hbar J_\alpha, \quad [J^\gamma, J_i] = 0, \\ [J_i, J_\alpha^d] &= \frac{1}{2}\hbar(\sigma_i)_{\beta\alpha}J_\beta^d, \quad \{J_\alpha^d, J_\beta^d\} = -\frac{1}{2}\hbar(C\sigma_i)_{\alpha\beta}J_i, \quad \{J_\alpha, J_\beta^d\} = -\frac{1}{4}\hbar C_{\alpha\beta}J^\gamma. \end{aligned} \quad (\text{B.2})$$

We next study how the above commutation relations are deformed in the presence of the supermonopole background (45). The contribution of the gauge field is added by making the replacements (52) in (B.1):

$$\begin{aligned} \Lambda^\gamma &= -\frac{2}{r}C_{\alpha\beta}\theta_\alpha\Lambda_\beta, \\ \Lambda_\alpha^d &= \frac{1}{r}(\sigma_i)_{\beta\alpha}(\theta_\beta\Lambda_i + x_i\Lambda_\beta). \end{aligned} \quad (\text{B.3})$$

They transform under the $osp(1, 2)$ transformation as

$$\begin{aligned} [L_i, \Lambda^\gamma] &= 0, \quad [L_\alpha, \Lambda^\gamma] = -\hbar\Lambda_\alpha^d, \\ [L_i, \Lambda_\alpha^d] &= \frac{1}{2}\hbar(\sigma_i)_{\beta\alpha}\Lambda_\beta^d, \quad \{L_\alpha, \Lambda_\beta^d\} = -\frac{1}{4}\hbar C_{\alpha\beta}\Lambda^\gamma, \end{aligned} \quad (\text{B.4})$$

where L_i and L_α are given in (55). Since we have added the contribution of the gauge field, Λ_α^d and Λ^γ no longer satisfy the $osp(2|2)$ algebra. We find that total angular momentum operators which satisfy the $osp(2|2)$ algebra are

$$\begin{aligned} L^\gamma &= -\frac{2}{r}C_{\alpha\beta}\theta_\alpha L_\beta - 4\hbar S, \\ L_\alpha^d &= \frac{1}{r}(\sigma_i)_{\beta\alpha}(\theta_\beta L_i + x_i L_\beta), \end{aligned} \quad (\text{B.5})$$

which are related to Λ^γ and Λ_α^d as

$$\begin{aligned} L^\gamma &= \Lambda^\gamma - \frac{2\hbar S}{r^2}(\theta C\theta) - 4\hbar S, \\ L_\alpha^d &= \Lambda_\alpha^d + \frac{\hbar S}{r^2}(\sigma_i)_{\beta\alpha}(\theta_\beta x_i + x_i\theta_\beta). \end{aligned} \quad (\text{B.6})$$

They transform under the $osp(1|2)$ transformation as

$$\begin{aligned} [L_i, L^\gamma] &= 0, \quad [L_\alpha, L^\gamma] = -\hbar L_\alpha^d, \\ [L_i, L_\alpha^d] &= \frac{1}{2}\hbar(\sigma_i)_{\beta\alpha}L_\beta^d, \quad \{L_\alpha, L_\beta^d\} = -\frac{1}{4}\hbar C_{\alpha\beta}L^\gamma. \end{aligned} \quad (\text{B.7})$$

An $OSp(1, 2)$ invariant quantity can be constructed from L^γ and L_α^d as

$$C_{\alpha\beta}L_\alpha^d L_\beta^d + \frac{1}{4}(L^\gamma)^2. \quad (\text{B.8})$$

By replacing L^γ and L_α^d with Λ^γ and Λ_α^d respectively, we can construct another $OSp(1, 2)$ invariant quantity. These are related as

$$C_{\alpha\beta}L_\alpha^d L_\beta^d + \frac{1}{4}(L^\gamma)^2 = C_{\alpha\beta}\Lambda_\alpha^d \Lambda_\beta^d + \frac{1}{4}(\Lambda^\gamma)^2 + 4S^2. \quad (\text{B.9})$$

In the last part of this section, we comment on the Hamiltonian. The Hamiltonian we analyzed in the section 4 actually does not provide a complete form of the kinetic term of a particle moving on $S^{2,2}$. It was written only by the $osp(1|2)$ generators. A complete Hamiltonian is constructed by using both of the $osp(1|2)$ and $osp(2|2)$ generators.

Let us consider the following two $osp(1|2)$ invariant quantities:

$$\begin{aligned} H_1 &= \frac{1}{2mr^2} (J_i J_i + C_{\alpha\beta} J_\alpha J_\beta), \\ H_2 &= \frac{1}{2mr^2} \left(C_{\alpha\beta} J_\alpha^d J_\beta^d + \frac{1}{4} J^2 \right), \end{aligned} \quad (\text{B.10})$$

where H_1 is the Hamiltonian used in the section 4. Considering the following replacements

$$-i \frac{\partial}{\partial x_i} \rightarrow p_i, \quad -i \frac{\partial}{\partial \theta_\alpha} \rightarrow (Cp)_\alpha, \quad (\text{B.11})$$

we rewrite the above Hamiltonians as

$$\begin{aligned} H_1 &= \frac{1}{2mr^2} \left[\left(x_i x_i + \frac{1}{4} (\theta C \theta) \right) p_j p_j + \frac{1}{4} \left(x_i x_i - \frac{1}{2} (\theta C \theta) \right) (p C p) + \frac{3}{2} i \epsilon_{ijk} x_j p_k (\theta \sigma_i C p) \right], \\ H_2 &= \frac{1}{2mr^2} \left[-\frac{1}{4} (\theta C \theta) p_j p_j - \frac{1}{8} (2r^2 + \theta C \theta) (p C p) + \frac{1}{2} i \epsilon_{ijk} x_j p_k (\theta \sigma_i C p) \right]. \end{aligned} \quad (\text{B.12})$$

We have used the following two relations,

$$\begin{aligned} x_i x_i + C_{\alpha\beta} \theta_\alpha \theta_\beta &= r^2, \\ x_i p_i + C_{\alpha\beta} \theta_\alpha p_\beta &= 0. \end{aligned} \quad (\text{B.13})$$

Therefore the following linear combination realizes the kinetic term of a particle on $S^{2,2}$:

$$H_1 - 3H_2 = \frac{1}{2mr^2} (p_i p_i + C_{\alpha\beta} p_\alpha p_\beta). \quad (\text{B.14})$$

We see that this combination can be the Hamiltonian without the gauge field.

C The representation theory of $OSp(1|2)$ and $OSp(2|2)$

In this section, we review the representation theory of $OSp(1|2)$ and $OSp(2|2)$. See [38, 39, 40] for references.

We denote the $osp(1|2)$ generators by $\{l_i, v_\alpha\}$, where $i = 1, 2, 3$ and $\alpha = 1, 2$. The bosonic part forms the $su(2)$ algebra. The $osp(1, 2)$ algebra is given by

$$[l_i, l_j] = i \epsilon_{ijk} l_k, \quad [l_i, v_\alpha] = \frac{1}{2} (\sigma_i)_{\beta\alpha} v_\beta, \quad \{v_\alpha, v_\beta\} = \frac{1}{2} (C \sigma_i)_{\alpha\beta} l_i. \quad (\text{C.1})$$

The irreducible representation of $OSp(1|2)$ is characterized by an integer or half-integer l which is called superspin. This representation is decomposed into the spin l and $(l - 1/2)$ representations of $SU(2)$. The dimension is $(2l + 1) + 2l = 4l + 1$. The quadratic Casimir is given by

$$l_i l_i + C_{\alpha\beta} v_\alpha v_\beta = l \left(l + \frac{1}{2} \right). \quad (\text{C.2})$$

We next consider the $OSp(2|2)$ group. Let $\{l_i, v_\alpha, d_\alpha, \gamma\}$ ($i = 1, 2, 3, \alpha = 1, 2$) be a basis of the $osp(2|2)$ algebra forming

$$\begin{aligned} [l_i, l_j] &= i\epsilon_{ijk}l_k, \quad [l_i, v_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}v_\beta, \quad \{v_\alpha, v_\beta\} = \frac{1}{2}(C\sigma_i)_{\alpha\beta}l_i, \\ [\gamma, l_\alpha] &= d_\alpha, \quad [\gamma, d_\alpha] = v_\alpha, \quad [\gamma, l_i] = 0, \\ [l_i, d_\alpha] &= \frac{1}{2}(\sigma_i)_{\beta\alpha}d_\beta, \quad \{d_\alpha, d_\beta\} = -\frac{1}{2}(C\sigma_i)_{\alpha\beta}l_i, \\ \{v_\alpha, d_\beta\} &= -\frac{1}{4}C_{\alpha\beta}\gamma. \end{aligned} \tag{C.3}$$

The bosonic part of the $osp(2|2)$ algebra forms $su(2) \oplus u(1)$ subalgebra whose generators are $\{l_i, \gamma\}$. The $osp(2|2)$ algebra contains the $osp(1|2)$ subalgebra $\{l_i, v_\alpha\}$ and has the automorphism such as

$$\{l_i, v_\alpha, d_\alpha, \gamma\} \rightarrow \{l_i, v_\alpha, -d_\alpha, -\gamma\}. \tag{C.4}$$

The $osp(2|2)$ algebra has two Casimir invariants:

$$\begin{aligned} C_2 &= (l_i^2 + C_{\alpha\beta}v_\alpha v_\beta) - (C_{\alpha\beta}d_\alpha d_\beta + \frac{1}{4}\gamma^2), \\ C_3 &= \frac{1}{2}\gamma C_2 + \frac{1}{2}\gamma C_{\alpha\beta}(v_\alpha v_\beta - d_\alpha d_\beta) + \frac{1}{3}(\sigma_i C)_{\alpha\beta}(-v_\alpha l_i d_\beta + d_\alpha l_i v_\beta) \\ &\quad + \frac{1}{6}(\sigma_i C)_{\alpha\beta}(-v_\alpha d_\beta + d_\alpha v_\beta)l_i. \end{aligned} \tag{C.5}$$

We summarize the irreducible representations of $osp(2|2)$. They are classified into two categories. One is called typical representation and the other non-typical representation. The typical representation is reducible with respect to $osp(1|2)$ and is not specified by the two Casimirs of $osp(2|2)$ since both of them vanish. On the other hand, the non-typical representation is irreducible with respect to $osp(1|2)$ and is specified by the two Casimirs of $osp(2|2)$. Any representations of $osp(2|2)$ are reducible with respect to $u(1) \oplus su(2)$ and are constructed by the direct sum of irreducible representations of $u(1) \oplus su(2)$. We label the representations by $(g; j, j_3)$:

$$\begin{aligned} l_i^2 |g; j, j_3\rangle &= j(j+1) |g; j, j_3\rangle, \\ l_3 |g; j, j_3\rangle &= j_3 |g; j, j_3\rangle, \\ \gamma |g; j, j_3\rangle &= 2g |g; j, j_3\rangle, \end{aligned} \tag{C.6}$$

where $j = 0, \frac{1}{2}, 1, \dots$, $j_3 = -j, -j+1, \dots, j$ and g takes an arbitrary complex number.

The irreducible representations of $osp(2|2)$ are classified into the following four cases.

$(g; 0)$

This is a trivial one-dimensional representation.

$(j; j)$

This is a $4j + 1$ dimensional representation and is decomposed into

$$|j; j, j_3\rangle \oplus |j + 1/2; j - 1/2, j_3\rangle. \tag{C.7}$$

It is a non-typical representation since it is irreducible with respect to $osp(1|2)$. $\{d_\alpha, \gamma\}$ are constructed from $\{l_i, l_\alpha\}$ as

$$\begin{aligned}\gamma &= \frac{1}{j + \frac{1}{4}} \left(C_{\alpha\beta} v_\alpha v_\beta + 2j \left(j + \frac{1}{2} \right) \right), \\ d_\alpha &= [\gamma, l_\alpha] = -\frac{1}{2 \left(j + \frac{1}{4} \right)} (\sigma_i)_{\beta\alpha} (v_\beta l_i + l_i v_\beta).\end{aligned}\tag{C.8}$$

Since both of the Casimirs (C.5) vanish

$$C_2 = C_3 = 0,\tag{C.9}$$

they do not specify the irreducible representation.

$(-j; j)$

This case is related to the $(j; j)$ representation by the automorphism of $osp(2, 2)$ (C.4) since the sign of $g = j$ changes when we change the sign of γ . It is also a non-typical and $4j + 1$ dimensional representation:

$$|-j; j, j_3\rangle \oplus |-j - 1/2; j - 1/2, j_3\rangle.\tag{C.10}$$

This representation is what was used in the appendix B.

$(g; j)$ ($2g \neq j$)

This representation is $8j$ ($j \neq 0$) dimensional one, which is decomposed into

$$|g; j, j_3\rangle \oplus |g + 1/2; j - 1/2, j_3\rangle \oplus |g - 1/2; j - 1/2, j_3\rangle \oplus |g, j - 1, j_3\rangle.\tag{C.11}$$

The first two representations form the superspin j irreducible representation of $osp(1, 2)$, while the last two do the superspin $j - \frac{1}{2}$ irreducible representation. Thus this representation is a typical one:

$$(g; j)_{osp(2,2)} \rightarrow (j)_{osp(1,2)} \oplus (j - 1/2)_{osp(1,2)}.\tag{C.12}$$

The Casimirs are given by

$$C_2 = j^2 - g^2, \quad C_3 = g(j^2 - g^2).\tag{C.13}$$

D Derivation of (54)

In this section, we show the detailed derivation of the equation (54). Commutation relations of Λ_i and Λ_α are calculated as

$$\begin{aligned}[\Lambda_i, \Lambda_j] &= i\hbar \epsilon_{ijk} \left(\Lambda_k + \tilde{\Lambda}_k^{(1)} \right), \\ [\Lambda_i, \Lambda_\alpha] &= \frac{1}{2} \hbar (\sigma_i)_{\beta\alpha} \left(\Lambda_\beta + \tilde{\Lambda}_\beta \right), \\ \{\Lambda_\alpha, \Lambda_\beta\} &= \frac{1}{2} \hbar (C\sigma_i)_{\alpha\beta} \left(\Lambda_i + \tilde{\Lambda}_i^{(2)} \right).\end{aligned}\tag{D.1}$$

where

$$\begin{aligned}
\tilde{\Lambda}_i^{(1)} &= \frac{1}{2}\epsilon_{jkl}x_kx_lF_{jl} + \frac{i}{2}x_j(\theta\sigma_j)_\alpha F_{i\alpha} - \frac{i}{2}x_i(\theta\sigma_j)_\alpha F_{j\alpha} + \frac{1}{8}\epsilon_{ijk}(\theta\sigma_jF\sigma_k^T\theta), \\
\tilde{\Lambda}_\alpha &= -\frac{1}{3}\epsilon_{ijk}(\theta\sigma_l\sigma_i)_\alpha x_jF_{kl} + \frac{1}{3}\epsilon_{ijk}(C\sigma_i\sigma_l)_{\alpha\beta}x_jx_lF_{k\beta}, \\
&\quad + \frac{i}{6}x_i(\theta FC\sigma_i)_\alpha - \frac{i}{3}\theta_\alpha x_i \text{tr}(FC\sigma_i) \\
\tilde{\Lambda}_i^{(2)} &= -\frac{1}{4}(\theta C\theta)\epsilon_{ijk}F_{jk} - \frac{i}{2}(\sigma_k\sigma_i\sigma_j)_{\alpha\beta}\theta_\alpha x_jF_{\beta k} + \frac{i}{4}\text{tr}(C\sigma_iF)x_jx_j - \frac{i}{2}\text{tr}(C\sigma_jF)x_jx_i. \quad (\text{D.2})
\end{aligned}$$

These are greatly simplified after we substitute the values of the field strength (45):

$$\tilde{\Lambda}_i^{(1)} = \tilde{\Lambda}_i^{(2)} = -\frac{eg}{r}x_i, \quad \tilde{\Lambda}_\alpha = -\frac{eg}{r}\theta_\alpha. \quad (\text{D.3})$$

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